CANONICAL BASIS AND HOMOLOGY OF LOCAL SYSTEMS

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Introduction

Let $U_q\mathfrak{sl}_2$ be the quantum group corresponding to the Lie algebra $\mathfrak{sl}_2$, and let M_λ be the Verma module over the $U_q\mathfrak{sl}_2$ with the highest weight λ. It follows from the results of [V1] that one can identify the weight space $M[\mu] = (M_{\lambda_1}\otimes\ldots\otimes M_{\lambda_n})[\mu], \mu=\sum\lambda_i - 2l$ with a suitable homology space of the configuration space $X_{n,l}=\{\mathbf{x}\in\mathbb{C}^l | x_i\ne x_j, x_i\ne 0, x_i\ne z_k, k=1,\ldots,n\}$ with coefficients in a certain one-dimensional local system. This result plays a crucial role in the construction of quantum group symmetries in conformal field theory, since this homology space naturally appears in the integral formulas for the solutions of Knizhnik-Zamolodchikov equations (see [SV1, SV2]).

On the other hand, for any simple Lie algebra $\mathfrak{g}$, Lusztig has defined a remarkable basis in a tensor product of irreducible finite-dimensional representations of $U_q\mathfrak{g}$, which he called "the canonical basis"(see [L, Chapter 27]). This basis is defined by two conditions:

(1) $\Psi(b)=b$, where Ψ is a certain antilinear involution involving the R-matrix.

(2) The canonical basis is related with the usual basis, given by the tensor product of Poincare-Birkhoff-Witt bases, by a matrix, all off-diagonal entries of which belong to $q^{-1}\mathbb{Z}[q^{-1}]$.

This basis generalizes the canonical basis in one irreducible finite-dimensional module over $U_q\mathfrak{g}$, also introduced by Lusztig, and has a number of remarkable properties (see [L]).

Note that the canonical basis in a tensor product is non-trivial even for $U_q\mathfrak{sl}_2$, since the definition of Ψ involves the quantum R-matrix. For this reason, the first condition is usually much more difficult to verify than the second one.

The goal of this paper is to combine these two results and give a geometric construction of the canonical basis in terms of the homologies of local systems (for technical reasons, we are constructing the basis dual to the canonical basis rather than the canonical basis itself). In this paper, we only consider $\mathfrak{g}=\mathfrak{sl}_2$ and assume that q is not a root of unity. In this case we are able to give an explicit construction of the dual canonical basis in every weight subspace of a tensor product of irreducible finite-dimensional $U_q\mathfrak{sl}_2$-modules (see Theorem 4.3). This answer is especially simple if all the highest weights are large enough compared to the level $l=(\sum\lambda_i-\mu)/2$ of the considered weight subspace. In this case, the basis dual to the canonical basis is given (up to some simple factor) by the bounded connected

components in the complement to certain hyperplanes in $\mathbb{R}^l$ (Corollary 4.7). Our construction is quite parallel to the algebraic constructions in [FK].

Notice that in [V2] a geometric construction of the crystal base in the space of singular vectors of $M[\mu]$ is given in terms of the same local system and critical points of the associated multivalued holomorphic function. It would be interesting to establish a direct connection between these two constructions.

Generalization of the results of this paper to arbitrary simple Lie algebras is more complicated; for example, even for $\mathfrak{sl}_3$ the natural generalization of the approach in [FK] fails to produce the canonical basis (see [KK]). We plan to address these questions in forthcoming papers.

1. $U_q\mathfrak{sl}_2$: NOTATIONS

Let the quantum group $U_q\mathfrak{sl}_2$ be the Hopf algebra over the field $\mathbb{C}(q^{\pm\frac{1}{2}})$ with generators $e, f, q^{\pm h}$ and commutation relations

$$
\begin{aligned}
q^h e &= q^2 e q^h,\\
q^h f &= q^{-2} f q^h,\\
[e,f] &= \frac{q^h - q^{-h}}{q - q^{-1}}.
\end{aligned} \tag{1.1}
$$

The comultiplication and antipode are given by

$$
\begin{gathered}
\Delta e = e \otimes q^{h/2} + q^{-h/2} \otimes e,\\
\Delta f = f \otimes q^{h/2} + q^{-h/2} \otimes f,\\
\Delta q^h = q^h \otimes q^h,\\
Se = -qe, \qquad Sf = -q^{-1} f, \qquad Sq^h = q^{-h}.
\end{gathered} \tag{1.2}
$$

This coincides with the definition in [V1, 4.1] if we make the following substitutions: $q \mapsto q^{\frac{1}{2}}, e \mapsto e/\sqrt{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, f \mapsto f/\sqrt{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$.

We will also use another generators of the same algebra, namely:

$$
E = q^{h/2} e, \qquad F = f q^{-h/2}. \tag{1.3}
$$

They satisfy the same commutation relations (1.1) as e, f, and the comultiplication is given by

$$
\begin{aligned}
\Delta E &= E \otimes q^h + 1 \otimes E,\\
\Delta F &= F \otimes 1 + q^{-h} \otimes F.
\end{aligned} \tag{1.4}
$$

Also, it is useful to note that

$$
\begin{aligned}
E^m &= q^{m(m+1)/2} e^m q^{mh/2},\\
F^m &= q^{m(m-1)/2} f^m q^{-mh/2}
\end{aligned} \tag{1.5}
$$

The universal R-matrix for $U_q\mathfrak{sl}_2$ is given by

$$\mathcal{R} = C\Theta, \quad C = q^{h\otimes h/2},$$

$$\Theta = \sum_{k\geq 0} q^{-k(k+1)/2}\frac{(q-q^{-1})^k}{[k]!}q^{kh/2}e^k \otimes q^{-kh/2}f^k \tag{1.6}$$
$$= \sum_{k\geq 0} q^{k(k-1)/2}\frac{(q-q^{-1})^k}{[k]!}E^k\otimes F^k.$$

Here, as usual, $[n]=\frac{q^n-q^{-n}}{q-q^{-1}}$, and $[n]! = [1][2]\dots[n]$.

For every pair V, W of finite-dimensional representations, the R-matrix gives rise to the commutativity isomorphism $\check{\mathcal{R}} = P\mathcal{R} : V\otimes W \to W\otimes V$, where P is the permutation: $P(v\otimes w) = w\otimes v$.

We shall only consider $U_q\mathfrak{sl}_2$-modules with weight decomposition: $V = \bigoplus_{\lambda\in\mathbb{C}} V[\lambda]$, and $q^h|_{V[\lambda]} = q^\lambda \operatorname{Id}$ (for a non-integer λ, this requires an appropriate extension of the field of scalars).

We denote by M_λ the Verma module with highest weight λ. For $\lambda\in\mathbb{Z}_+$ this module has a finite-dimensional quotient:

$$V_\lambda = M_\lambda/(f^{\lambda+1}v_\lambda).$$

Then V_λ is an irreducible finite-dimensional module of dimension $\lambda+1$, and every irreducible finite-dimensional module with weight decomposition is isomorphic to one of V_λ.

Finally, define an algebra anti-automorphism $\tau : U_q\mathfrak{sl}_2 \to U_q\mathfrak{sl}_2$ by

$$\tau(e) = f, \quad \tau(f) = e, \quad \tau(q^h) = q^h, \quad \tau(ab) = \tau(b)\tau(a). \tag{1.7}$$

Then τ is a coalgebra automorphism: $(\tau\otimes\tau)\Delta(x) = \Delta(\tau(x))$, and $\tau(\mathcal{R}) = \mathcal{R}^{21}$. From now on, we will also denote by τ the map $\tau\otimes\dots\otimes\tau : (U_q\mathfrak{sl}_2)^{\otimes n} \to (U_q\mathfrak{sl}_2)^{\otimes n}$.

For every module M let the contragredient module M^c be the restricted dual to M with the action of $U_q\mathfrak{sl}_2$ given by

$$\langle gv^*, v\rangle = \langle v^*, \tau(g)v\rangle, \qquad v\in M, \quad v^*\in M^c, \quad g\in U_q\mathfrak{sl}_2. \tag{1.8}$$

Note that we have a canonical isomorphism $(M_1\otimes M_2)^c \simeq M_1^c\otimes M_2^c$ and that for $\lambda\in\mathbb{Z}_+$, $V_\lambda^c\simeq V_\lambda$. Also, for every λ we have a canonical morphism $M_\lambda\to M_\lambda^c$, and the image of this morphism is exactly the irreducible highest-weight module L_λ; in particular, for $\lambda\in\mathbb{Z}_+$, the image is V_λ. Thus, we have an embedding

$$V_\lambda \subset M_\lambda^c. \tag{1.9}$$

Combining this embedding with the canonical pairing $M_\lambda\otimes M_\lambda^c\to\mathbb{C}(q^{\pm\frac{1}{2}})$, we get a non-singular bilinear form on V_λ, which is usually called the contragredient (or Shapovalov) form.

All the theory above can be as well developed when q is a complex number rather than a formal variable, provided that q is not a root of unity. We will use the same notations in this case.

Remark 1.1. Our notations differ slightly from those of Lusztig. They are related as follows:

Ours	Lusztig's
q	v
E	F
F	E
q^h	K^{-1}
Θ	$\bar{\Theta}$

(in the last line, $\bar{\ }$ is the bar involution, see (3.1)).

2. Homology of local systems

Fix the following data:
– a positive integer n
– a collection of weights $\lambda_1, \dots, \lambda_n \in \mathbb{C}$
– points $z_1, \dots, z_n \in \mathbb{R}, 0 < z_1 < \dots < z_n$
– an integer $l \in \mathbb{Z}_+$
– a number $\kappa \in \mathbb{R} \setminus \mathbb{Q}$
We denote $\lambda = \lambda_1 + \dots + \lambda_n$ and

$$q = e^{\pi \mathrm{i}/\kappa}.$$

Note that the number q defined by the formula above is not a root of unity.

Let $\mathcal{C} \subset \mathbb{C}^l = \{(x_1, \dots, x_l)\}$ be the configuration of hyperplanes defined by the equations

$$\begin{gathered} x_i = x_j, \quad i, j = 1 \dots l, \\ x_i = z_k, \quad i = 1, \dots, l, \quad k = 1, \dots, n, \\ x_i = 0, \quad i = 1, \dots, l. \end{gathered} \tag{2.1}$$

Let $Y = \mathbb{C}^l \setminus \mathcal{C}$, and let $\mathcal{S}$ be the one-dimensional complex local system over Y such that its flat sections are $s(x) = const$(univalent branch of $\psi(x)$), where $const \in \mathbb{C}$ and

$$\psi = \prod_{i<j} (x_i - x_j)^{2/\kappa} \prod_{i,p} (x_i - z_p)^{-\lambda_p/\kappa} \prod_{p<q} (z_p - z_q)^{\lambda_p \lambda_q / 2\kappa}. \tag{2.2}$$

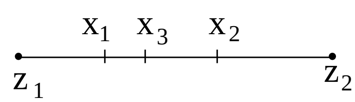
Let Σ_l be the symmetric group acting on $\mathbb{C}^l$ by permutation of coordinates; this action preserves $\mathcal{C}$ and $\mathcal{S}$. We will be interested in the relative homology group

$H_l(\mathbb{C}^l;\mathcal{C};\mathcal{S})$ with coefficients in $\mathcal{S}$. We have a natural action of Σ_l on this homology space. Let us for brevity denote by H_l the antisymmetric part of the homology:

$$H_l = H_l^{-\Sigma_l}(\mathbb{C}^l;\mathcal{C};\mathcal{S}). \tag{2.3}$$

Sometimes we will also write $H_l(0, z_1, \dots, z_n; 0, \lambda_1, \dots, \lambda_n)$ to emphasize the dependence on z_i, λ_i (the first zero is to remind of the point 0 also used in the definition; it will be needed later when we define slightly more general type of homologies). In fact, the space $H_l(0, \mathbf{z}; 0, \boldsymbol{\lambda})$ is defined for any $\mathbf{z} \in X_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n | z_i \neq z_j, z_i \neq 0\}$, and $H_l(0, \mathbf{z}; 0, \boldsymbol{\lambda})$ form a vector bundle over X_n. Moreover, this bundle has a natural flat connection (Gauss-Manin connection, see [V1]). However, unless otherwise noted, we will only use $\mathbf{z}$ such that $0 < z_1 < \dots < z_n$.

We will frequently use pictures to represent cycles in $H_l(\mathbb{C}^l;\mathcal{C};\mathcal{S})$. For example, we will represent the cycle $z_1 \leq x_1 \leq x_3 \leq x_2 \leq z_2$ by the picture



We will choose a section s of the local system over such a cycle as follows. For each pair of indices i, j define

$$Br((x_i - x_j)^\alpha) = \begin{cases} e^{\alpha \mathrm{Log}(x_i - x_j)}, & Re(x_i) > Re(x_j), \\ e^{\alpha \mathrm{Log}(x_j - x_i)}, & Re(x_i) < Re(x_j), \end{cases}$$

where $\mathrm{Log}(x)$ is the main branch of the logarithm defined for $Re\ x > 0$ by the condition $\mathrm{Log}\ x \in \mathbb{R}_+$ for $x > 0$. This definition is chosen so that for $x_i, x_j \in \mathbb{R}_+, \alpha \in \mathbb{R}$, $Br(x_i - x_j)^\alpha \in \mathbb{R}_+$. Similarly, define

$$Br(\psi) = \prod Br(x_i - x_j)^{2/\kappa} \prod Br(x_i - z_k)^{-\lambda_k/\kappa} \prod Br(z_i - z_j)^{\lambda_i \lambda_j / 2\kappa}.$$

One easily sees that if a region $D \subset \mathbb{C}^l$ is such that $x_i - x_j \notin \mathrm{i}\mathbb{R}, x_i - z_k \notin \mathrm{i}\mathbb{R}$ on D, then $Br(\psi)$ is a section of the local system $\mathcal{S}$ on D. In particular, if we have a simplex $c \subset \mathbb{C}^l$ such that the order of $Re\ x_i, Re\ z_i$ is fixed on c then $Br(\psi)$ defines a section of $\mathcal{S}$ over c. In [V1], this section is called "the positive branch of ψ".

We will also use more elaborate l-dimensional cycles, such as the one used in formula (2.6) below. It will be convenient to give the following definition; it may seem complicated but is in fact quite natural.

Definition 2.1. Assume that we have fixed the data $n, \mathbf{z}, \boldsymbol{\lambda}, l, \kappa$ as at the beginning of this section. Let us additionally assume that $Re\ z_i \neq Re\ z_j, Re\ z_i \neq 0$. A "combinatorial cycle" C is a figure in $\mathbb{C}$ consisting of a finite number of intervals of smooth curves ("arcs") such that:

- these arcs do not intersect
- each arc can intersect a vertical line at most at one point
- the endpoints of each arc are either some of the specified points z_i or 0, and the endpoints of the same arc can not coincide
- on each arc, there are some marked points so that the total number of the marked points on all arcs is equal to l.

- these marked points are labeled by $x_1, \dots, x_l$

For every combinatorial cycle C as above, we define the corresponding relative cycle in $H_l(\mathbb{C}^l, \mathcal{C}; \mathbb{Z})$ by the following rule. Consider the space $\mathbb{R}^l$ with the coordinates $x_1, \dots, x_l$ and the standard orientation. Let Δ_C be the subset in $\mathbb{R}^l$ given by the following conditions:

- $0 \le x_i \le 1$
- if the points x_i and x_j are marked on the same arc of C, and x_i is to the left of x_j then $x_i \le x_j$ for all points in Δ_C.

Obviously, Δ_C is a direct product of simplices.

Now, let us number all the arcs of C in an arbitrary way and let $\gamma_k : [0,1] \to \mathbb{C}$ be a parameterization of the k-th arc such that Re $\gamma(t)$ is a strictly increasing function. Define the map $\gamma : \Delta_C \to \mathbb{C}^l$ by

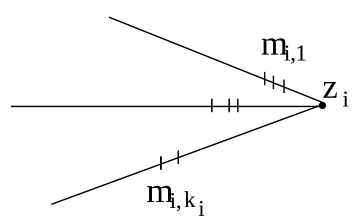
$$\gamma(x_1, \dots, x_l) = (\gamma_{k_1}(x_1), \dots, \gamma_{k_l}(x_l)),$$

where k_i is the index of the arc on which the point x_i is marked. This defines an l-dimensional chain in $\mathbb{C}^l$, which we will also denote by C. Informally, this chain can be described by saying that the points x_i run along the corresponding arcs preserving the order.

Define a section s of the local system $\mathcal{S}$ over C by the rule

$$s(\mathbf{x}) = q^{\sum a_i} Br(\psi(\mathbf{x})),$$

where $\mathbf{x}$ is such a point that all x_i are close to the right end of the corresponding arc, and the order of $Re\ x_i$ is chosen as in the following picture:



and $a_i = -\sum_{1 \le a < b \le k_i} m_{i,a} m_{i,b} b$

Lemma 2.2. *Let C be a combinatorial cycle. Then the pair (C, s) defines an element of the relative homology space $H_l(\mathbb{C}^l, \mathcal{C}; \mathcal{S})$, which only depends on the homotopy type of the arcs and the order of the points placed on these arcs. These cycles are flat sections of the bundle $H_l(\mathbb{C}^l, \mathcal{C}; \mathcal{S})$ over X_n with respect to the Gauss-Manin connection.*

The reason for calling these cycles "combinatorial" is that they are defined by a finite collection of data. Later we will show that such cycles span the whole homology space H_l (see Theorem 2.3).

Note that the action of a permutation $\sigma \in \Sigma_l$ on the combinatorial cycles is given by permuting the indices of the points x_i and multiplying by $(-1)^{|\sigma|}$ (the sign comes from the change of orientation). Since we are only interested in the antisymmetric part of the homology, from now on we won't put the labels x_i on the points. Instead, we will just indicate the number of points on each arc, assuming summation over all $l!$ possible labelings, which automatically gives an anti-symmetric cycle.

Let M_λ be a Verma module over $U_q\mathfrak{sl}_2$ with $q = e^{\pi \mathrm{i}/\kappa}$ and let v_λ be a highest weight vector. Vectors $F^{(m)}v_\lambda = \frac{F^m v_\lambda}{[m]!}, m = 0, 1, \ldots$ form a basis in M_λ. Denote by $(F^{(m)}v_\lambda)^*$ the dual basis in M^c_λ. More generally, for any n-tuple $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n_+$ denote

$$F^{(\mathbf{m})} = F^{(m_1)}v_{\lambda_1} \otimes \ldots \otimes F^{(m_n)}v_{\lambda_n} \in M_{\lambda_1} \otimes \ldots \otimes M_{\lambda_n}. \tag{2.4}$$

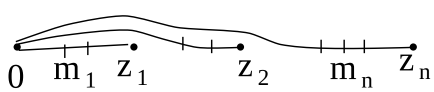
These monomials form a basis in $M_{\lambda_1} \otimes \ldots \otimes M_{\lambda_n}$. Let

$$(F^{(\mathbf{m})})^* = (F^{(m_1)}v_{\lambda_1})^* \otimes \ldots \otimes (F^{(m_n)}v_{\lambda_n})^* \tag{2.5}$$

be the dual basis in $M^c_{\lambda_1} \otimes \ldots \otimes M^c_{\lambda_n}$. It is easy to see that $(F^{(\mathbf{m})})^* \in M^c_{\lambda_1} \otimes \ldots \otimes M^c_{\lambda_n}[\sum \lambda_i - 2\sum m_i]$ (recall that for any $U_q\mathfrak{sl}_2$-module V we denote by $V[\mu]$ the subspace of vectors of weight μ in V).

Theorem 2.3. *Assume that $z_i \in \mathbb{R}, 0 < z_1 < z_2 \cdots < z_n$. Define the map $\varphi_{\mathbf{z}} : M^c_{\lambda_1} \otimes \ldots \otimes M^c_{\lambda_n}[\lambda - 2l] \to H_l$ by*

(2.6)

$(F^{(\mathbf{m})})^* \mapsto [m_1]! \ldots [m_n]!$ 

where $l = \sum m_i$.

Then:

1. *Map (2.6) is an isomorphism.*
2. *For any $i = 1, \ldots, n-1$, we have the following commutative diagram:*

$$\begin{array}{ccc} M^c_{\lambda_1} \otimes \ldots \otimes M^c_{\lambda_n}[\lambda - 2l] & \xrightarrow{\varphi_{\mathbf{z}}} & H_l(0, \mathbf{z}; 0, \boldsymbol{\lambda}) \\ \Big\downarrow \check{\mathcal{R}}_i & & \Big\downarrow T_i \\ M^c_{\lambda_1} \otimes \ldots \otimes M^c_{\lambda_{i+1}} \otimes M^c_{\lambda_i} \otimes \ldots \otimes M^c_{\lambda_n}[\lambda - 2l] & \xrightarrow{\varphi_{s_i(\mathbf{z})}} & H_l(0, s_i(\mathbf{z}), 0, s_i(\boldsymbol{\lambda})) \end{array}$$

where $s_i(\mathbf{z}) = (z_1, \ldots, z_{i+1}, z_i, \ldots, z_n), s_i(\boldsymbol{\lambda}) = (\lambda_1, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_n)$ and T_i is the monodromy along the path shown on Figure 1.

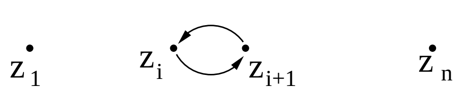


FIGURE 1

Proof. This theorem follows from the results of [V1]. For the sake of completeness, we quote here the statement in the most convenient for us form. Let $z_0 < \cdots < z_n \in \mathbb{R}$, and let $\mathcal{C}(z_0, \ldots, z_n)$ be a configuration of hyperplanes in $\mathbb{C}^l$, given by the

("weights") and let $\mathcal{S}$ be the one-dimensional local system on $\mathbb{C}^l \setminus \mathcal{C}(z_0, \ldots, z_n)$ defined as in (2.2), but with the indices p, q running from 0 to n rather than from 1 to n. Denote by $H_l(z_0, \ldots, z_n; \lambda_0, \ldots, \lambda_n) = H_l^{-\Sigma_l}(\mathbb{C}^l, \mathcal{C}(z_0, \ldots, z_n), \mathcal{S})$ the antisymmetric part of the relative homology space. Obviously, if we let $z_0 = 0, \lambda_0 = 0$ then this space coincides with previously defined $H_l = H_l(0, z_1, \ldots, z_n; 0, \lambda_1, \ldots, \lambda_n)$.

Recall that a vector v in a $U_q\mathfrak{sl}_2$-module V is called singular if $ev = 0$. We denote by V^{sing} the subspace of singular vectors in [V1]. Then the following result is a corollary of Theorem 5.11.13 in [V1] (see also Figure 5.17).

Theorem 2.4 ([V1]). *Let $z_0, \ldots, z_n \in \mathbb{R}, z_0 < \cdots < z_n$. There exists an isomorphism*

$$\varphi : \left(M^c_{\lambda_0} \otimes \ldots \otimes M^c_{\lambda_n}\right)^{sing}[\sum_0^n \lambda_i - 2l] \simeq H_l(z_0, \ldots, z_n; \lambda_0, \ldots, \lambda_n) \tag{2.7}$$

such that

(1) *Let $C_{m_1,\ldots,m_n} \in H_l(z_0, \ldots, z_n; \lambda_0, \ldots, \lambda_n)$ be the cycle in the right-hand side of* (2.6) (*with* 0 *replaced by* z_0). *Then*

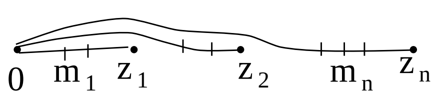
$$\varphi^{-1}(C_{m_1,\ldots,m_n}) = v^*_{\lambda_0} \otimes (F^{(m_1)}v_{\lambda_1})^* \otimes \ldots \otimes (F^{(m_n)}v_{\lambda_n})^* + \ldots,$$

where dots stand for a combination of monomials of the form $(F^{(m_0)}v_{\lambda_0})^* \otimes \ldots \otimes (F^{(m_n)}v_{\lambda_n})^*$ *with* $m_0 > 0$.

(2) *For any $i = 0, \ldots, n-1$, we have a commutative diagram similar to that in Theorem* 2.3 *part* (2).

Remark. In [V1], the basis $f^m = f^{m_1}v_{\lambda_1} \otimes \ldots \otimes f^{m_n}v_{\lambda_n}$ is used instead of $F^{(m)}$. In these notations, formula (2.6) takes the form

$$(f^{m_1}v_{\lambda_1})^* \otimes \ldots \otimes (f^{m_n}v_{\lambda_n})^* \mapsto q^{a(\mathbf{m})}$$



where

$$a(\mathbf{m}) = \sum_{i=1}^n \frac{m_i(m_i - 1 - \lambda_i)}{2}.$$

(compare with [V1, 5.2.22])

In order to get our Theorem 2.3 from Theorem 2.4, we also need the following lemma.

Lemma 2.5. *Let V be an arbitrary highest weight module over $U_q\mathfrak{sl}_2$, and let $M^c_{\lambda_0}$ be the contragredient Verma module with the highest weight λ_0. Then: for every homogeneous vector $v \in V[\mu]$ there exists a unique vector $\tilde{v} \in (M^c_{\lambda_0} \otimes V)^{sing}[\lambda_0 + \mu]$ such that*

$$\tilde{v} = v^*_{\lambda_0} \otimes v + \ldots,$$

where dots stand for a combination of vectors of the form $(F^{(m)}v_{\lambda_0})^* \otimes v'$ with $m > 0$.

This gives us an isomorphism

$$\begin{aligned} V[\mu] &\simeq (M^c_{\lambda_0}\otimes V)^{sing}[\lambda_0+\mu] \\ v &\mapsto \tilde{v} \end{aligned} \tag{2.8}$$

Proof of the Lemma. Follows from the following identities:

$$\begin{aligned}(M^c_{\lambda_0}\otimes V)^{sing}[\lambda_0+\mu] =& \operatorname{Hom}_{U_q\mathfrak{b}}(\mathbb{C}_{\lambda_0+\mu}, M^c_{\lambda_0}\otimes V) = \operatorname{Hom}_{U_q\mathfrak{b}}((M^c_{\lambda_0})^*\otimes \mathbb{C}_{\lambda_0+\mu}, V)\\ =& \operatorname{Hom}_{U_q\mathfrak{b}}((M^c_{-\mu})^*, V) = V[\mu],\end{aligned}$$

where $U_q\mathfrak{b}$ is the subalgebra in $U_q\mathfrak{sl}_2$ generated by q^h and e and $\mathbb{C}_{\lambda_0+\mu}$ is the one-dimensional module over $U_q\mathfrak{b}$ with the action given by $e=0, q^h=q^{\lambda_0+\mu}$. The last identity follows from the fact that $(M^c_{-\mu})^*$ is nothing but the lowest weight Verma module with lowest weight μ and thus is free over $U_q\mathfrak{n}^+$. □

Now we can easily prove our Theorem 2.3. Indeed, letting in Theorem 2.4 $\lambda_0 = 0, z_0=0$, we get an isomorphism $H_l \simeq (M^c_0\otimes M^c_{\lambda_1}\otimes\ldots\otimes M^c_{\lambda_n})^{sing}[\lambda-2l]$. On the other hand, by Lemma 2.5 this is isomorphic to $M^c_{\lambda_1}\otimes\ldots\otimes M^c_{\lambda_n}[\lambda-2l]$. Combining these isomorphisms, we get an isomorphism $H_l\simeq M^c_{\lambda_1}\otimes\ldots\otimes M^c_{\lambda_n}[\lambda-2l]$. It is easy to see that this isomorphism is given by formula (2.6), and satisfies the required properties. □

We will also need a modification of this theorem. As before, let $z_0,\ldots,z_n\in\mathbb{R}, z_0<\cdots<z_n$, and let $\lambda_0=0$. Then we have an embedding of the unions of hyperplanes: $\mathcal{C}(z_1,\ldots,z_n)\subset\mathcal{C}(z_0,z_1,\ldots,z_n)$, which induces a map of homologies

$$i: H_l(z_1,\ldots,z_n;\lambda_1,\ldots,\lambda_n)\to H_l(z_0,\ldots,z_n;0,\lambda_1,\ldots,\lambda_n).$$

Theorem 2.6. *Denote for brevity $M^c = M^c_{\lambda_1}\otimes\ldots\otimes M^c_{\lambda_n}$. Then the following diagram is commutative:*

$$\begin{tikzcd} (M^c)^{sing}[\lambda-2l] \arrow[r, "\varphi"] \arrow[d] & H_l(z_1,\ldots,z_n;\lambda_1,\ldots,\lambda_n) \arrow[d, "i"'] \\ M^c[\lambda-2l]\simeq(M^c_0\otimes M^c)^{sing}[\lambda-2l] \arrow[r, "\varphi"] & H_l(z_0,\ldots,z_n;0,\lambda_1,\ldots,\lambda_n) \end{tikzcd} \tag{2.9}$$

where the first vertical line is the natural embedding $(M^c)^{sing}[\lambda-2l]\subset M^c[\lambda-2l]$.

This theorem immediately implies the following corollaries, proof of which is trivial:

Corollary 2.7. *1. The map $i: H_l(z_1,\ldots,z_n;\lambda_1,\ldots,\lambda_n)\to H_l(z_0,\ldots,z_n;0,\lambda_1,\ldots,\lambda_n)$ is an embedding.*

2. Let $z_0=0$, and let $C\in H_l$ be a combinatorial cycle such that 0 is not an endpoint of any of the arcs. Then the corresponding vector $\varphi^{-1}(C)\in(M^c_{\lambda_1}\otimes\ldots\otimes M^c_{\lambda_n})[\lambda-2l]$ is singular.

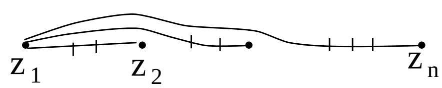
Proof of Theorem 2.6. First, note that the composition of maps $(M^c)^{sing}[\lambda-2l]\hookrightarrow M^c[\lambda-2l]\to(M^c_0\otimes M^c)^{sing}[\lambda-2l]$, where the last arrow is the isomorphism of

Lemma 2.5, is given by $v \mapsto v_0^* \otimes v$, which immediately follows from the definition. Similarly, the embedding $M^c[\lambda-2l] \hookrightarrow (M^c \otimes M_0^c)^{sing}[\lambda-2l]$ is given by $v \mapsto v \otimes v_0^*$.

Next, let $z_1 < \cdots < z_n < z_0$ (note this change of order!). Denote for brevity $\mathbf{z} = (z_1, \ldots, z_n)$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. Then we have the following commutative diagram:

$$\begin{tikzcd} (M^c)^{sing}[\lambda-2l] \arrow[r, "\varphi"] \arrow[d] & H_l(\mathbf{z};\boldsymbol{\lambda}) \arrow[d] \\ M^c[\lambda-2l] \simeq (M^c \otimes M_0^c)^{sing}[\lambda-2l] \arrow[r, "\varphi"] & H_l(\mathbf{z}, z_0; \boldsymbol{\lambda}, 0) \end{tikzcd} \tag{2.10}$$

Indeed, for cycles $C \in H_l(\mathbf{z}, \boldsymbol{\lambda})$ of the form shown below, (2.10) immediately follows from the definition of φ in Theorem 2.4. On the other hand, these cycles span the homology space $H_l(\mathbf{z}, \boldsymbol{\lambda})$, which follows from Theorem 2.4.



Now we need to change the order of the points. Denote by $\check{R}$ the intertwining operator $PR_{M_0^c, M^c} : M_0^c \otimes M^c \to M_c \otimes M_0^c$. It can be written as a product of R-matrices, each interchanging M_0^c with one of the factors $M_{\lambda_i}^c$. By Theorem 2.4(2), we have the following commutative diagram:

$$\begin{tikzcd} (M^c \otimes M_0^c)^{sing}[\lambda-2l] \arrow[r, "\varphi"] \arrow[d, "\check{R}^{-1}"] & H_l(\mathbf{z}, z_0; \boldsymbol{\lambda} 0) \arrow[d, "T^{-1}"'] \\ (M_0^c \otimes M^c)^{sing}[\lambda-2l] \arrow[r, "\varphi"] & H_l(z_0, \mathbf{z}; 0, \boldsymbol{\lambda}) \end{tikzcd} \tag{2.11}$$

where the operator $T : H_l(z_0, \mathbf{z}; 0, \boldsymbol{\lambda}) \to H_l(\mathbf{z}, z_0; \lambda_1, \boldsymbol{\lambda}, 0)$ is the monodromy along the path shown on Figure 2.

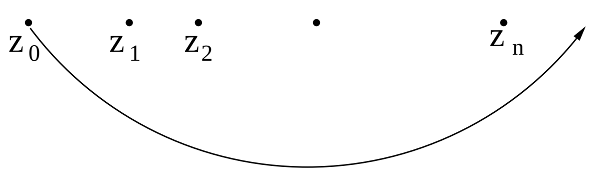


FIGURE 2

Combining (2.10) and (2.11), we get a commutative diagram as in the statement of the theorem. It remains to check that composition of the maps $(M^c)^{sing} \subset (M^c \otimes M_0^c)^{sing} \xrightarrow{\check{R}^{-1}} (M_0^c \otimes M^c)^{sing}$ coincides with the embedding $(M^c)^{sing} \subset (M_o^c \otimes M^c)^{sing}$ in (2.9). This follows from $\check{R}(v_0^* \otimes v) = v \otimes v_0^*$, which is immediate from the definition of $\mathcal{R}$. Similarly, the composition of the maps

$$H_l(\mathbf{z};\boldsymbol{\lambda}) \to H_l(\mathbf{z}, z_0; \boldsymbol{\lambda}, 0) \xrightarrow{T^{-1}} H_l(z_0, \mathbf{z}; 0, \boldsymbol{\lambda})$$

is equal to the map of homology spaces in (2.9): it suffices to check this for a combinatorial cycle in $H_l(\mathbf{z};\boldsymbol{\lambda})$ in which case it is obvious. □

3. Complex conjugation and canonical basis

As before, we assume that we have fixed data $n, l, \kappa, \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n), \mathbf{z} = (z_1, \dots, z_n)$ as in the beginning of Section 2. from now on, we assume in addition that $\lambda_i \in \mathbb{R}$. Note also that since we assumed $\kappa \in \mathbb{R}$, we have $\bar{q} = q^{-1}$. From now on, we will note need the dependence of the homology space on the points z_i, so we will just write H_l for the homology space defined by (2.3). We will identify H_l with the weight subspace $M^c_{\lambda_1} \otimes \dots \otimes M^c_{\lambda_n}[\lambda - 2l]$ using the isomorphism of Theorem 2.3.

Let $\overline{} : U_q\mathfrak{sl}_2 \to U_q\mathfrak{sl}_2$ be the antilinear algebra automorphism defined by

$$\overline{F} = F, \quad \overline{E} = E, \quad \overline{q^h} = q^{-h} \tag{3.1}$$

Define the associated antilinear involution on the Verma module M_λ by $\overline{xv_\lambda} = \bar{x}v_\lambda, x \in U_q\mathfrak{sl}_2$. More generally, if M_1, M_2 are highest weight modules then define an antilinear involution $\psi : M_1 \otimes M_2 \to M_1 \otimes M_2$ by

$$\psi(v \otimes v') = \overline{\Theta}(\bar{v} \otimes \overline{v'}), \tag{3.2}$$

where Θ is defined by (1.6). One can check that $\psi^2 = 1$.

Theorem 3.1 (Lusztig). *Let V, V' be irreducible finite-dimensional $U_q\mathfrak{sl}_2$-modules with the highest-weight vectors v_λ, v_μ respectively. Let $b_i = F^{(i)}v_\lambda, b'_i = F^{(i)}v_\mu$ be the bases in V, V'. Then there exists a unique basis $b_{ij} = b_i \diamondsuit b'_j$ in $V \otimes V'$ such that:*

1. *$b_i \diamondsuit b'_j = b_i \otimes b'_j + \sum_{k \neq 0} a_k b_{i-k} \otimes b'_{j+k}$, and $a_k \in q^{-1}\mathbb{Z}[q^{-1}]$*
2. *$\psi(b_i \diamondsuit b'_j) = b_i \diamondsuit b'_j$.*

Moreover, the coefficients a_k in (1) are non-zero only for $k > 0$.

The basis introduced above is called the canonical basis and has many remarkable properties, which can be found in [L].

We will also need the dual basis. Let M be a highest-weight module, and M^c the corresponding contragredient module. Define the involution on M^c by $\langle \overline{v^*}, v \rangle = \overline{\langle v^*, \bar{v} \rangle}$. As before, let $b^*_k = (F^{(k)}v_\lambda)^*$ be the basis in M^c dual to the basis $b_k = F^{(k)}v_\lambda$ in M. Since for $\lambda \in \mathbb{Z}_+$ we have an embedding $V_\lambda \subset M^c_\lambda$, the elements $b^*_k, 0 \leq k \leq \lambda$ also give a basis in V_λ, dual to the basis b_k with respect to the contragredient form.

Theorem 3.2. *Let V, V' be irreducible finite-dimensional $U_q\mathfrak{sl}_2$-modules. Denote by $(b_i \diamondsuit b'_j)^*$ the basis in $V \otimes V'$ dual to the canonical basis in $V \otimes V'$. Then this basis has the following properties, which uniquely determine it:*

(1)

$$(b_i \diamondsuit b'_j)^* = b^*_i \otimes {b'_j}^* + \sum_{k>0} \alpha_k b^*_{i+k} \otimes {b'_{j-k}}^*, \quad \alpha_k \in q^{-1}\mathbb{Z}[q^{-1}] \tag{3.3}$$

(2) *$\psi^c((b_i \diamondsuit b'_j)^*) = (b_i \diamondsuit b'_j)^*$, where*

$$\psi^c(v^* \otimes (v')^*) = \tau(\Theta)(\overline{v^*} \otimes \overline{(v')^*}). \tag{3.4}$$

Here Θ, C *are defined by* (1.6).

Proof. The theorem immediately follows from the definitions. $\square$

From now on, we will call the basis, dual to the canonical basis, "the dual canonical basis". There are explicit formulas for the canonical basis and the dual canonical basis – see [FK].

More generally, one can define the canonical basis (and therefore, the dual canonical basis) in a tensor product of any number of irreducible finite-dimensional representations. This construction is also due to Lusztig; we will use it in a more explicit form, taken from [FK].

Let $V_1, \dots, V_n$ be irreducible finite-dimensional representations. For every $i = 1, \dots, n-1$ let $\check{\mathcal{R}}_i : V_1 \otimes \dots \otimes V_n \to V_1 \otimes \dots \otimes V_{i+1} \otimes V_i \otimes \dots \otimes V_n$ be the operator $\check{\mathcal{R}} = P\mathcal{R}$ acting on $V_i \otimes V_{i+1}$. Let σ_0 be the longest element in the symmetric group Σ_n: $\sigma_0 : (1 \dots n) \mapsto (n \dots 1)$, and let $\sigma_0 = s_{i_1} \dots s_{i_l}$ be its reduced expression. Define

$$\check{\mathcal{R}}^{(n)} = \check{\mathcal{R}}_{i_1} \dots \check{\mathcal{R}}_{i_l} : V_1 \otimes \dots \otimes V_n \to V_n \otimes \dots \otimes V_1. \tag{3.5}$$

Proposition 3.3.

(1) $\check{\mathcal{R}}^{(n)}$ *does not depend on the choice of the reduced expression for* σ_0.

(2) $\check{\mathcal{R}}^{(n)} = \sigma_0 \mathcal{R}^{(n)}$, *where* $\mathcal{R}^{(n)}$ *is defined inductively by* $\mathcal{R}^{(2)} = \mathcal{R}$,

$$\mathcal{R}^{(n)} = (1 \otimes \mathcal{R}^{(n-1)}) \cdot (1 \otimes \Delta^{n-2})(\mathcal{R}) = (\mathcal{R}^{(n-1)} \otimes 1) \cdot (\Delta^{n-2} \otimes 1)(\mathcal{R})$$

and $\sigma_0 : V_1 \otimes \dots \otimes V_n \to V_n \otimes \dots \otimes V_1$ *is the permutation.*

(3) *Let*

$$\begin{aligned} C^{(n)} &= (1 \otimes C^{(n-1)}) \cdot (1 \otimes \Delta^{n-2})(C) = (C^{(n-1)} \otimes 1) \cdot (\Delta^{n-2} \otimes 1)(C) = q^{\frac{1}{2}\sum_{i<j} h_i \otimes h_j}, \\ \Theta^{(n)} &= (1 \otimes \Theta^{(n-1)}) \cdot (1 \otimes \Delta^{n-2})(\Theta) = (\Theta^{(n-1)} \otimes 1) \cdot (\Delta^{n-2} \otimes 1)(\Theta) \end{aligned}$$

Then $\mathcal{R}^{(n)} = C^{(n)} \Theta^{(n)}$.

(4)
$$\tau(\Theta^{(n)}) = \check{\mathcal{R}}^{(n)} (C^{(n)})^{-1} \sigma_0.$$

Proof. (1) follows from the Yang-Baxter equation; (2) and (3) are proved in [FK, 1.2]. To prove (4), note that the obvious identities $\tau(\mathcal{R}) = \mathcal{R}^{21}, \tau(C) = C$ and the fact that τ is a coalgebra automorphism imply $\tau(\mathcal{R}^{(n)}) = \sigma_0(\mathcal{R}^{(n)}) = \sigma_0 \cdot \mathcal{R}^{(n)} \cdot \sigma_0, \tau(C^{(n)}) = C^{(n)}$, after which we can use $\mathcal{R}^{(n)} = C^{(n)} \Theta^{(n)}$. $\square$

Now we define the dual canonical basis in a tensor product of n representations:

Proposition 3.4 (see [FK]). *Let* $V_1, \dots, V_n$ *be irreducible finite dimensional representations, and let* $(F^{(\mathbf{m})})^*$ *be the dual monomial basis* (2.5) *in* $V_1 \otimes \dots \otimes V_n$. *Then there exists a unique basis* $b_{\mathbf{m}}$ *in* $V_1 \otimes \dots \otimes V_n$ *such that*

(1)
$$b_{\mathbf{m}} = (F^{(\mathbf{m})})^* + \sum a_{\mathbf{k}} (F^{(\mathbf{k})})^*, \quad a_{\mathbf{k}} \in q^{-1}\mathbb{Z}[q^{-1}].$$

(2) $\psi^c(b_{\mathbf{m}}) = b_{\mathbf{m}}$, *where*

$$\psi^c(v_1 \otimes \ldots \otimes v_n) = \tau(\Theta^{(n)})(\overline{v_1} \otimes \ldots \otimes \overline{v_n}). \tag{3.6}$$

This basis is called the dual canonical basis.

Remark. Similar to the case $n = 2$, Lusztig defines a canonical basis in a tensor product of n modules (see [L, Chapter 27]). One can easily prove that the basis defined in Proposition 3.4 is dual to the canonical basis of Lusztig. However, it is not needed for our purposes.

The main goal of our paper is to describe the dual canonical basis in terms of the homology space H_l defined by (2.3).

Lemma-Definition 3.5. *Define an antilinear map $C_k(\mathbb{C}^l, \mathcal{C}, ; \mathcal{S}) \to C_k(\mathbb{C}^l, \mathcal{C}; \mathcal{S})$ as follows: if Δ is a singular simplex, s a section of the local system $\mathcal{S}$ over Δ, then let $(\Delta, s) \mapsto (\bar{\Delta}, \overline{s(\bar{z})})$, where a bar denotes the standard complex conjugation in $\mathbb{C}^l$. Then this map of complexes induces an antilinear involution on H_l which will be denoted by $\overline{}$.*

The first main result of this paper is the following theorem:

Theorem 3.6. *Under the assumptions of the previous Lemma, the isomorphism $M^c_{\lambda_1} \otimes \ldots \otimes M^c_{\lambda_n}[\lambda - 2l] \simeq H_l$ constructed in Theorem* 2.3 *identifies the complex conjugation in H_l with the involution ψ^c defined by* (3.6)

Proof. It suffices to check it on the basis elements $(F^{(\mathbf{m})})^*$, where it is proved by direct calculation using Theorem 2.3 and the identity $\tau(\Theta^{(n)}) = \check{\mathcal{R}}^{(n)}(C^{(n)})^{-1}\sigma_0$ (see Proposition 3.3(4)). $\square$

Remark 3.7. In fact, an obvious analog of Theorem 3.6 holds for any simple Lie algebra $\mathfrak{g}$.

4. Explicit construction of the dual canonical basis.

In this section we give a geometric construction of the canonical basis in a tensor product of irreducible representations. This construction is a geometric counterpart of the algebraic construction in [FK].

Let $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}_+$. Denote for brevity

$$M = M_{\lambda_1} \otimes \ldots \otimes M_{\lambda_n}, \quad V = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}. \tag{4.1}$$

Then we have an embedding

$$V \subset M^c. \tag{4.2}$$

We give a geometric construction of the dual canonical basis in the weight subspace $V[\lambda - 2l] \subset M^c[\lambda - 2l] \simeq H_l$. Let $\lambda_i \in \mathbb{Z}_+$ be such that $\sum \lambda_i \geq l$. Let B_l be the set of all combinatorial cycles (see Lemma-definition 2.1) in the space H_l such that:

(4.3)
(1) On each arc, there is exactly one point x placed.
(2) All the arcs are in the upper half plane.
(3) For each i, z_i serves as an endpoint for at most λ_i arcs.
(4) If for some i, z_i serves as an endpoint for less than λ_i arcs then there are no arcs passing over z_i.

An example of a cycle $b \in B_l$ is given on Figure 3; here $n = 6, l = 7$ and $\lambda_1 = 3, \lambda_2 = 2, \lambda_3 \geq 4, \lambda_4 = \lambda_5 = 1, \lambda_6 \geq 1$.

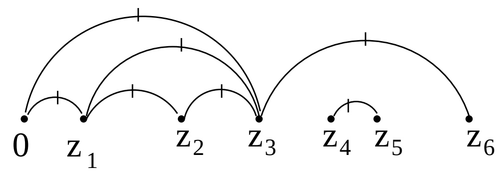


FIGURE 3. An example of a cycle $b \in B_l$.

Proposition 4.2. *For every $b \in B_l$ define $\mathbf{a}(b) = (a_1, \ldots, a_n) \in \mathbb{Z}_+^n$ by the rule $a_i =$ (number of arcs connecting z_i with all $z_j, j < i$). Then $b \mapsto \mathbf{a}(b)$ is a bijection between B_l and the set*

$$P_{\boldsymbol{\lambda}}(l) = \{\mathbf{a} \in \mathbb{Z}_+^n | a_i \leq \lambda_i, \sum a_i = l\} \tag{4.4}$$

Proof. This proposition is of purely combinatorial nature, and can be proved by induction on l using the following easily verified fact: if we remove from $b \in B_l$ the extreme right arc then we get a cycle from B_{l-1}. Conversely, given a cycle $b \in B_l$ and an index $i, 1 \leq i = len$ there exists at most one way to add an arc to b so that the right end of this arc is z_i, and so that the resulting cycle lies in B_{l+1}. □

This proposition allows us to index the elements of B_l by $\mathbf{m} \in P_{\boldsymbol{\lambda}}(l) \subset \mathbb{Z}_+^n$: we will write $b = b_{\mathbf{m}}$ if b is an element of B_l such that $\mathbf{a}(b) = \mathbf{m}$.

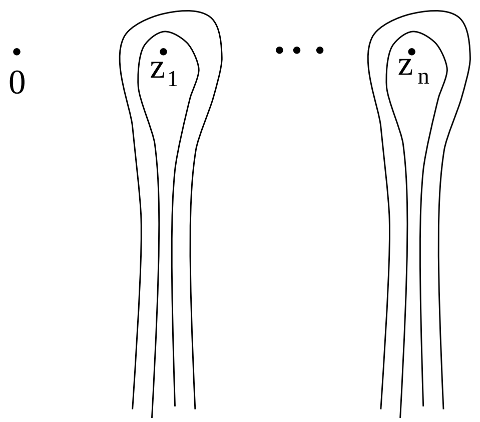
Theorem 4.3. *The elements $b_{\mathbf{m}} \in B_l, \mathbf{m} \in P_{\boldsymbol{\lambda}}(l)$ form the dual canonical basis in $V[\lambda - 2l] \subset M^c[\lambda - 2l] \simeq H_l$.*

Note that in the combinatorial construction of the dual canonical basis in [FK], the basis is also parameterized by the pictures of the form (4.3). However, the interpretation of these pictures is different there.

The remaining part of this section and the two subsequent sections are devoted to the proof of this theorem. The proof consists of several steps.

Proposition 4.4. *Every $b \in B_l$ lies in V.*

Proof. Consider the following finite collection of infinite l-dimensional chains C in Y:

$C_{k_1,\dots,k_n} =$ 

where there are k_1 loops around z_1, k_2 loops around z_2, and so on, with $\sum k_i = l$. Fix a section of the local system $\mathcal{S}^*$ over each chain. Then each such a chain C defines a linear functional on H_l by the rule $f \mapsto \langle f, C\rangle$, where $\langle , \rangle$ is the intersection pairing. One easily sees that this functional is well-defined.

Due to Theorem 2.3, one can choose some sections over $C_{k_1,\dots,k_n}$ so that the chains $C_{k_1,\dots,k_n}$ will form a basis dual to the monomial basis (2.5) in M^c. Thus, $v^* \in V \subset M^c$ if and only if $\langle v^*, C_{k_1,,\dots,k_n}\rangle = 0$ for every $(k_1,\dots,k_n)$ such that $k_i > \lambda_i$ for some i.

It follows from (4.3) that every $b \in B_l$ satisfies this condition and thus, $b \in V$. $\square$

The crucial step in proving Theorem 4.3 is proving the next two propositions.

Proposition 4.5. *For every $b = b_{\mathbf{m}} \in B$,*

$$b_{\mathbf{m}} = (F^{(\mathbf{m})})^* + \sum_{\mathbf{k}>\mathbf{m}} c_{\mathbf{k}} (F^{(\mathbf{k})})^*,$$

where $(F^{(\mathbf{m})})^*$ is the monomial basis (2.5)*, $c_{\mathbf{k}} \in q^{-1}\mathbb{Z}[q^{-1}]$ and $<$ is the lexicographic order on $\mathbb{Z}_+^n$.*

Remark. In fact, it follows from $b \in V$ that the sum above can only contain monomials $(F^{(\mathbf{k})})^*$ for $\mathbf{k}$ such that $k_i \le \lambda_i, \sum k_i = l$.

Proposition 4.6. *Every $b \in B_l$ is real, i.e. $\overline{b} = b$.*

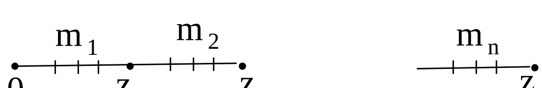
The proofs of Propositions 4.5, 4.6 will be given in Sections 5 and 6.

Now we can easily prove Theorem 4.3. Propositions 4.4, 4.5 imply that elements of B lie in V and are linearly independent. On the other hand, it is obvious from Proposition 4.2 that the number of elements in B_l is equal to the dimension of $V[\lambda - 2l]$. Therefore, B_l is a basis in $V[\lambda - 2l]$. Propositions 4.5 and 4.6 show that this basis satisfies the definition of the dual canonical basis. $\square$

This theorem has many corollaries.

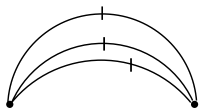
Corollary 4.7. *Assume that $\lambda_i \ge l$ for all i. Then the dual canonical basis is given by*

(4.5)

$b_{\mathbf{m}} = [m_1]! \dots [m_n]!$ 

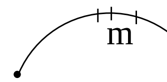
Proof. Immediately follows from Theorem 4.3 and the following lemma, which is easily proved by induction:

Lemma 4.8.



m arcs

$= [m]!$



□

Note that the cycles on the right-hand side of (4.5) are exactly the anti-symmetric combinations of the bounded connected components of $\mathbb{R}^l \setminus \mathcal{C}_{\mathbb{R}}$. It has been known for a long time that the connected components form a basis in the relative homology space $H_L(\mathbb{C}^l, \mathcal{C}; \mathcal{S})$ for any local system $\mathcal{S}$ (see ([A], [S] and references therein). Therefore, the theorem above shows that the anti-symmetrization of this basis for the local system described above has a natural interpretation in terms of representation theory of $U_q\mathfrak{sl}_2$: up to the factor $[m_1]! \dots [m_n]!$, it is nothing else but the dual canonical basis in $V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}$ under the assumptions that $\lambda_i \geq l$ for all i.

Another important special case is the dual canonical basis in the subspace of singular vectors. Let us recall that a vector $v \in V$ is called singular ($v \in V^{sing}$) if $ev = 0$. It follows from the general result of Lusztig ([L, 27.2.5]) that there exists some subset $B^{sing} \subset B$ of the dual canonical basis which is a basis in the subspace V^{sing}. The following theorem describes this basis explicitly.

Theorem 4.9. *Let B^{sing} be the subset of all cycles $b \in B_l$ satisfying the following additional condition:*

$$\text{(4.6)} \qquad \textit{There are no arcs ending at } 0.$$

Then B^{sing} is a basis in the subspace $V^{sing} \subset V$.

Proof. By Corollary 2.7, we see that every $b \in B^{sing}$ is a singular vector. Using Lemma 2.5 and simple combinatorial arguments, similar to those used in the proof of Proposition 4.2, it is easy to show that the dimension of V^{sing} is equal to the number of elements in B^{sing}. □

In particular, if $\lambda_1 + \dots + \lambda_n - 2l = 0$ then Theorem 4.9 describes a basis in the space $(V[0])^{sing}$, which coincides with the space of invariants in V. Note that in this case condition (4.6) is equivalent to the following condition:

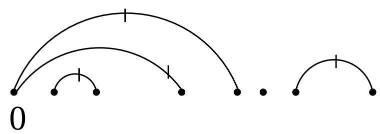
$$\text{(4.7)} \qquad \text{Each point } z_i \text{ is an endpoint for exactly } \lambda_i \text{ arcs.}$$

Note that if we complete the real line by adding the point ∞ so that the real line becomes a circle and the upper half-plane becomes a disk then the cycles $b \in B^{inv}$ correspond to all the ways to join n points on a circle by non-intersecting arcs with multiplicities so that the total number of arcs ending at z_i is equal to λ_i. Again, this parameterization of the basis in the subspace of invariants is completely parallel to the combinatorial construction in [FK]. Both the geometric construction presented here and the combinatorial construction in [FK] make the cyclic symmetry of the dual canonical basis in the space of invariants (see [L, 28.2.9]) obvious.

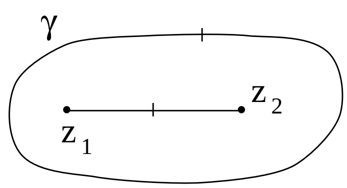
5. Proofs: the case of 2-dimensional modules.

In this section we give the proofs of Propositions 4.5, 4.6 in the special case where all $\lambda_i = 1$, so that all V_{λ_i} are two-dimensional. Later we will show how to reduce the general case to this special one.

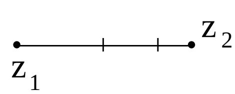
Throughout this section, we assume $\lambda_i = 1$ for all i. In this case, conditions (4.3) imply that for every cycle $b \in B_l$ and every i, the point z_i is an endpoint of at most one arc. Here is a typical example of a cycle $b \in B_l$:

(5.1) 

Lemma 5.1. *In the notations of Section 2, let $n = 2$. Then the cycle shown below represents the zero class in the anti-symmetrized homology space H_l.*

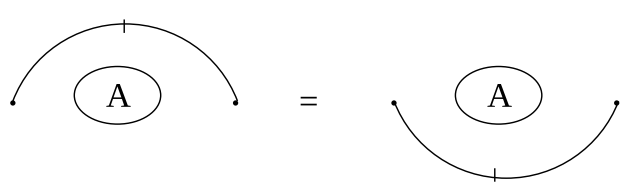
$C =$ 

Proof. Note first that it is indeed a cycle, since the monodromy around γ is equal to 1. Deforming this cycle so that γ approaches the interval $[z_1, z_2]$, we get that

$C = const(1+q^{-2}-1-q^{-2})$ 

□

Corollary 5.2. *Let $A \in H_l$ be any combinatorial cycle satisfying conditions* (4.3, 4.6)*. Then*

(5.2) 

Repeated application of this corollary yields the following result:

Proposition 5.3. *Every cycle $b \in B_l$ satisfies $\bar{b} = b$.*

This is nothing but Proposition 4.6 in our special case.

Lemma 5.4. *Let A be any combinatorial cycle satisfying conditions* (4.3, 4.6)*, and let B, C be any combinatorial cycles contained in the corresponding regions* (*dotted lines on the figure*)*. Then*

(5.3)

Proof. Obvious, since the "total weight" of the cycle A is equal to zero (that is, monodromy around the cycle encompassing A is trivial). $\square$

Using this proposition, we can prove by induction the following corollary.

Corollary 5.5. *Let $b \in B_l$. Then we have the following identity in $H_l \simeq M^c[\lambda - 2l]$:*

$$b = (F^{(\mathbf{a}(b))})^* + \sum_{\mathbf{k} > \mathbf{a}(b)} c_{\mathbf{k}} (F^{(\mathbf{k})})^*,$$

where $c_{\mathbf{k}} \in q^{-1}\mathbb{Z}[q^{-1}]$, $<$ is the lexicographic order on $\mathbb{Z}^n$ and $\mathbf{a}(b) \in \mathbb{Z}^n_+$ is defined by $a_i = 0$ if z_i is the left endpoint of some arc, and $a_i = 1$ otherwise.

This is exactly Proposition 4.5 in our special case, which concludes the proof of Theorem 4.3 for the case $\lambda_i = 1$.

6. Proof: the general case

In this section we will prove Propositions 4.5, 4.6 (and thus, Theorem 4.3) for arbitrary $\lambda_i \in \mathbb{Z}_+$. We reduce the general situation to the case $\lambda_i = 1$, which has been proved already. Our reduction is parallel to the approach in [FK].

Recall the setup of Section 3 and the anti-symmetrized homology space H_l.

Let us choose points $0 < \tilde{z}_1 < \cdots < \tilde{z}_\lambda$, $\lambda = \sum \lambda_i$, and consider the configuration of hyperplanes $\tilde{\mathcal{C}} \subset \mathbb{C}^l$ defined as in (2.1) with z_i replaced by $\tilde{z}_p, 1 \le p \le \lambda$ and n replaced by λ. Let $\tilde{\mathcal{S}}$ be the local system on $\mathbb{C}^l \setminus \tilde{\mathcal{C}}$ defined by (2.2) with the changes as above and with $\tilde{\lambda}_p = 1$. Denote by $\tilde{H}_l$ the corresponding anti-symmetrized homology space.

Theorem 6.1. *There exists a map $i_{\boldsymbol{\lambda}} : \tilde{H}_l \to H_l$ such that*

(1) *It preserves the complex conjugation* (*see Lemma-Definition* 3.5)
(2) *Let $\pi : \{1, \ldots, \lambda\} \to \{1, \ldots, n\}$ be defined by $\pi(i) = 1$ if $1 \le i \le \lambda_1$, $\pi(i) = 2$ if $\lambda_1 + 1 \le i \le \lambda_1 + \lambda_2$, and so on. Let $\tilde{C} \in \tilde{H}_l$ be any combinatorial cycle*

lying in the upper half-plane. Then: $i_{\boldsymbol{\lambda}}(\tilde{C}) = 0$ *if* $\tilde{C}$ *contains a curve connecting* $\tilde{z}_i$ *with* $\tilde{z}_j$ *such that* $\pi(i) = \pi(j)$*. Otherwise,* $C = i_{\boldsymbol{\lambda}}(\tilde{C})$ *is also a combinatorial cycle lying in the upper half-plane and defined as follows: to each curve in* $\tilde{C}$ *connecting* $\tilde{z}_i$ *with* $\tilde{z}_j$ *there corresponds a unique curve in* C *connecting* $z_{\pi(i)}$ *with* $z_{\pi(j)}$*, and with the same number of points* x *marked on it.*

Example. Let $n = 2, \lambda_1 = \lambda_2 = 2$. Then

Proof. Let us choose closed non-intersecting disks $D_1, \dots, D_n$ in $\mathbb{C}$ such that D_i encloses the points $\tilde{z}_k$ with $\pi(k) = i$ (see Figure 3). Choose a continuous map $\phi : \mathbb{C} \to \mathbb{C}$ such that

(1) $\phi(D_i) = z_i, \phi(0) = 0$
(2) ϕ gives a diffeomorphism $\mathbb{C} \setminus \{D_1, \dots, D_n\} \simeq \mathbb{C} \setminus \{z_1, \dots, z_n\}$.
(3) $\phi(\bar{z}) = \overline{\phi(z)}$

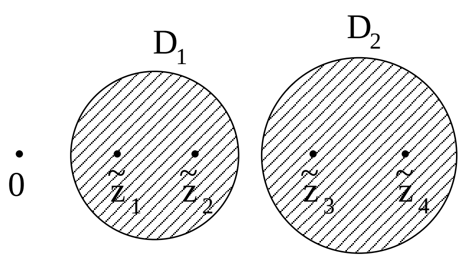


Figure 3.

It is easy to see that such a map exists. Let $\phi^l = \phi \times \dots \times \phi : \mathbb{C}^l \to \mathbb{C}^l$. Then $\phi(\tilde{\mathcal{C}}) = \mathcal{C}$, and thus ϕ induces a map of complexes of relative chains $C_k(\mathbb{C}^l, \tilde{\mathcal{C}}) \to C_k(\mathbb{C}^l, \mathcal{C})$.

To extend it to chains with coefficients in local systems, we need one more observation. Denote $U = \{\mathbf{x} \in \mathbb{C}^l | x_i \neq x_j, x_i \notin D_k, x_i \neq 0\}$. The following lemma can be easily obtained by comparing monodromies:

Lemma. *The following local systems on* U *are isomorphic:*

$$\tilde{\mathcal{S}}|_U \simeq ((\phi^l)^* \mathcal{S})|_U \tag{6.1}$$

Let us fix an isomorphism in (6.1) by the condition that it identifies $Br(\tilde{\psi})$ with $Br(\psi)$ for $0 \ll Re\ x_1 \ll \dots \ll Re\ x_l$.

Now we can extend the map of relative chains to the relative chains with coefficients in local systems: if $\tilde{\Delta}$ is a simplex in $\mathbb{C}^l$ and $\tilde{s}$ is a section of $\tilde{\mathcal{S}}$ over

$\Delta \cap (\mathbb{C}^l \setminus \tilde{\mathcal{C}})$ then let $\phi(\tilde{\Delta}, \tilde{s}) = (\Delta, s)$, where $\Delta = \phi(\tilde{\Delta})$ and s is the image of $\tilde{s}$ under the morphism $\Gamma(\tilde{\Delta} \cap \mathbb{C}^l \setminus \tilde{\mathcal{C}}, \tilde{\mathcal{S}}) \to \Gamma(\tilde{\Delta} \cap U, \tilde{\mathcal{S}}) \simeq \Gamma(\Delta \cap (\mathbb{C}^l \setminus \mathcal{C}), \mathcal{S})$. One easily verifies that this map is indeed a morphism of complexes and commutes with the action of the symmetric group Σ_l; thus, it gives rise to a map of homologies $i_{\boldsymbol{\lambda}} : \tilde{H}_l \to H_l$. It is easy to verify that so defined $i_{\boldsymbol{\lambda}}$ satisfies all the conditions of the theorem. (This uses the special choice of sections of the local system over combinatorial cycles, made in Definition 2.1.) □

Remark 6.2. It can be shown that if we rewrite the map $i_{\boldsymbol{\lambda}}$ using the identification $H_l \simeq M^c[\lambda - 2l]$ (see Theorem 2.3), then it coincides with the dual to the canonical embedding

$$M_{\lambda_1} \otimes \ldots \otimes M_{\lambda_n} \to M_1^{\otimes \lambda_1} \otimes M_1^{\otimes \lambda_2} \otimes \ldots \otimes M_1^{\otimes \lambda_n}$$

Corollary 6.3. *Propositions* 4.5, 4.6 *hold for arbitrary* $\lambda_i \in \mathbb{Z}_+$.

Proof. Using the map $i_{\boldsymbol{\lambda}}$ constructed in the previous theorem we see that every $b \in B$ can be written as $i_{\boldsymbol{\lambda}}(\tilde{b})$ for some $\tilde{b} \in \tilde{B}_l$. Since $i_{\boldsymbol{\lambda}}$ commutes with the complex conjugation, $\bar{b} = b$ follows from $\bar{\tilde{b}} = \tilde{b}$, which was proved in the previous section. Similarly, the triangularity condition follows from similar statement for $\tilde{b}$ (Corollary 5.5) and Lemma 4.8 allowing one to rewrite the monomial basis $(F^{(\mathbf{m})})^*$ in terms of the arcs with only one marked point.
□

Therefore, we have proved Theorem 4.3, which gives us a construction of the dual canonical basis in $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$ for arbitrary highest weights λ_i.

References

[A] Aomoto, K., *Gauss-Manin connection of integral of difference products*, J. Math. Soc. Japan **39** (1987), 191–207.

[FK] Frenkel, I. and Khovanov, M., *Canonical bases in tensor products and graphical calculus for* $U_q(\mathfrak{sl}_2)$, preprint, August 1995.

[KK] Khovanov, M. and Kuperberg, G., *Web bases for* $sl(3)$ *are not dual canonical*, Preprint, available from *http://www.math.ucdavis.edu/˜greg/*.

[L] Lusztig, G., *Introduction to quantum groups*, Birkhäuser, Boston, 1993.

[S] Salvetti, M., *Topology of the complement of real hyperplanes in* $\mathbb{C}^n$, Invent. Math. **88** (1987), 603–618.

[SV1] Schechtman, V. and Varchenko, A., *Arrangements of hyperplanes and Lie algebra homology*, Inv. Math. **106** (1991), 139–194.

[SV2] ———, *Quantum groups and homology of local systems*, Algebraic geometry and analytic geometry (Tokyo, 1990), ICM-90 Satell. Conf. Proc., Springer, Tokyo, 1991, pp. 182–197.

[V1] Varchenko, A., *Multidimensional hypergeometric functions and representation theory of quantum groups*, Adv. Ser. Math. Phys., vol. 21, World Scientific, River Edge, NJ, 1995.

[V2] ———, *Asymptotic solutions to the Knizhnik–Zamolodchikov equation and crystal base*, Comm. Math. Phys. **171** (1995), 99–138.